Indian Statistical Institute
B. Math. Hons. III Year & M. Math. I Year
Second Semester Examination 2008
Differential Geometry

Date: Marks: 100 Instructor: V. Pati

Attempt all questions. Each question carries 25 marks.

1. (a) Prove that $M = \{(x, y, z) : x^3 - y^2 + z^2 = 1\}$ is a smooth submanifold of $\mathbb{R}^3$, and compute its tangent space at $p = (0, 0, 1)$.

   (b) Let $f : M \to \mathbb{R}$ be the smooth map defined by $f(x, y, z) = y^2 + z^3$. Compute $Df(p)$ where $p = (0, 0, 1)$.

2. (a) Find the condition that the C^∞-vector field $X(x, y) = \alpha(x, y)\partial_x + \beta(x, y)\partial_y$ be tangent to $S^1 = \{(x, y) : x^2 + y^2 = 1\}$ at each point $(x, y) \in S^1$.

   (b) Find the condition that X as above is a left (= right) invariant vector field on S^1, assuming the condition found in (a) is satisfied. (**Note:** the group operation on S^1 is multiplication of unit complex numbers.)

3. (a) Let $M : \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1 + z^2\}$ be the hyperboloid of one-sheet in $\mathbb{R}^3$. Let $U = M \cap \{y = 0, x > 0\}$. Using suitable local coordinates on U, compute the 1st fundamental form (metric) on U induced from $\mathbb{R}^3$.

   (b) Compute the 2nd fundamental form of M above, and its mean and scalar curvatures.

4. (a) Consider the unit disc $\mathbb{D} = \{(x, y) : x^2 + y^2 < 1\}$ with the Poincare metric $g(x, y) = \frac{4(dx^2+dy^2)}{(1-x^2-y^2)^2}$. Compute its Christoffel symbols.

   (b) Compute the scalar curvature of the metric above.